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Semicontinuous Nonstationary Stochastic Games II

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In this paper we introduce a Borel space framework for zero-sum discrete-time stochastic games that is a game theoretic extension of some nonstationary dynamic programming models in the sense of Hinderer. Our game model, which allows for all of the primitive data to be nonstationary, contains a large class of Markov games. At the same time, it constitutes a considerable generalization of the game model introduced by Sengupta. To ensure the existence of a value of a nonstationary stochastic game and the existence of universally measurable optimal or ε-optimal strategies for the players we impose some asymmetric semicontinuity and compactness conditions on the primitive data that are weaker than those described in the existing literature on the subject. A certain special case, in which the players can restrict themselves to Borel measurable strategies, is also studied. © 1990 Academic Press, Inc.

1. INTRODUCTION

This paper deals with a Borel space framework for zero-sum discrete-time stochastic games in which all the primitive data are nonstationary. The transition law in our framework is a sequence of Borel measurable transition probabilities associating with every n-stage history of a game a probability distribution of the $(n+1)$th state and the payoff is the limit of a nondecreasing sequence of finite stage payoffs. Such a model is called nonstationary and is a natural extension of so-called stationary stochastic (or Markov) games introduced by Shapley [33] and subsequently studied by many authors (see [11–14, 17, 24], and their references). At the same time, it constitutes a game theoretic extension of some nonstationary dynamic programming models investigated by Hinderer [9], Schäl [26, 29], and Brown [6]. To ensure the existence and the universal (or Borel) measurability of a value function of a nonstationary stochastic game and the existence of universally (or Borel) measurable optimal or ε-optimal strategies for either or both players, we impose certain additional conditions on the model mentioned above. We give two sets of such conditions; see the game models BM_1 and BM_2 in Section 3. In BM_1, we make certain asymmetric assumptions on the primitive data, inspired by the Fan

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minimax theorem [8]. For example, the finite stage payoffs (the transition probabilities) are semicontinuous (strongly continuous) with respect to the actions of one player only, say the minimizer. Also the sets of admissible actions are assumed to be compact for this player only. For stationary stochastic games similar assumptions have been made in [17]. In BM_1, the players are allowed to use universally measurable strategies. This is motivated by the fundamental papers of Blackwell *et al.* [5] and Bertsekas and Shreve [3]. In BM_2, we present some symmetric compactness and semicontinuity conditions. Similar assumptions for stationary stochastic games have been made in [13, 14, 24]. The model BM_2 is, in fact, a special case of BM_1 (see Remark 3.1), but we prove that in BM_2 the players can restrict themselves to Borel measurable strategies.

Zero-sum nonstationary stochastic games have already been studied by Sengupta [32], Schäl [31], and Nowak [15, 19]. Sengupta has investigated in [32] a game model, suggested by Blackwell, in which the set of states is a compact metric space, the action sets are finite, the payoff is a lower semicontinuous function on the space of histories of the game, and the transition law is stationary. His game is an extension of an infinite game with incomplete information studied by Orkin in [22] and his proof refers to Blackwell's work [4]. Sengupta's game is a special case of BM_2 (see Remarks 3.1 and 3.2) while BM_1 constitutes a generalization of the game studied by the author (by means of different methods) in [15], where the state and action spaces are assumed to be countable sets. The papers of Nowak [19] and Schäl [31] present alternative approaches to zero-sum nonstationary stochastic games. In [19], for example, a different condition is imposed on the transition law. Namely, the transition probabilities are assumed to be weakly continuous. Schäl in [31] has assumed a certain kind of information lack, which is not considered here. For further bibliographic notes and some comments on the game model, our assumptions, and the results obtained, we refer the reader to Remarks 3.1–3.3, 4.1, and 5.1 and to [19].

This paper is organized as follows. Section 2 contains some preliminaries. The stochastic game model and main assumptions are described in Section 3. The main results are stated in Section 4. In Section 5 we prove two minimax selection theorems which are basic for this paper. Finally, in Section 6 we provide the proofs of the main results.

2. Preliminaries

In this section we introduce some notation and basic definitions. Let N denote the set of positive integers. We use R to denote the real line and we write R^* for $R \cup \{+\infty\}$. If X is a separable metric space, then we denote by $\mathscr{B}_X$ the σ-algebra of all Borel subsets of X. The symbol P_X represents the set of all probability measures on $\mathscr{B}_X$. We always assume that P_X is given the *weak topology* and the *Borel σ-algebra* $\mathscr{B}_{P_X}$ (see [3, Chap. 7] or [23]). It is known that if X is a separable (compact) metric space, then P_X is a separable (compact) metrizable space [3, Propositions 7.20 and 7.22]. A separable metric space X is called a *Borel space* or a *Borel set* if X is a Borel subset of a complete separable metric space and is endowed with the relative topology and the Borel σ-algebra $\mathscr{B}_X$. If X is a Borel space, then P_X is a Borel space as well [3, Corollary 7.25.1].

Let N^N be the set of all sequences of positive integers, endowed with the product topology. A separable metric space X is called an *analytic space* or an *analytic set* if there is a continuous mapping on N^N whose range is X (see [3, Chap. 7] or [10]). It is known that every Borel space is analytic [3, Proposition 7.36].

Let X be a Borel space. We denote by $\mathscr{U}_X$ the σ-algebra of all *universally measurable* subsets of X. The *limit σ-algebra*, denoted by $\mathscr{L}_X$, is the smallest σ-algebra containing the Borel subsets of X and closed under the Suslin operation (operation (A)). It is known that $\mathscr{B}_X \subset \mathscr{L}_X \subset \mathscr{U}_X$ [3, Chap. 7 and Appendix B].

Let X and Y be Borel spaces. A function $f\colon X \to Y$ is called *Borel* (*limit, universally*) *measurable* if $f^{-1}(B) \in \mathscr{B}_X$ ($f^{-1}(B) \in \mathscr{L}_X$, $f^{-1}(B) \in \mathscr{U}_X$) for every $B \in \mathscr{B}_Y$. Clearly, if f is limit measurable, then it is universally measurable. A function $f\colon X \to R^*$ is called *upper semianalytic* (u.s.a.) if the set $\{x \in X\colon f(x) > c\}$ (equivalently, the set $\{x \in X\colon f(x) \geq c\}$) is analytic for each $c \in R$. (A function $f\colon X \to R^*$ is called lower semianalytic if $-f$ is u.s.a. [3].) It is known that every Borel measurable function is u.s.a. and every u.s.a. function is universally measurable. We shall denote by $M(X)$ the *set of all bounded below universally measurable functions* $f\colon X \to R^*$.

Assume X and Y are Borel spaces. By a *Borel* (*limit, universally*) *measurable transition probability* from X to Y (or a *stochastic kernel* on Y given X) we mean a Borel (limit, universally) measurable mapping $f\colon X \to P_Y$. It is known that $f\colon X \to P_Y$ is a Borel (limit, universally) measurable transition probability from X to Y if and only if $f(\cdot)(B)$ is a Borel (limit, universally) measurable mapping from X into $[0, 1]$, for each $B \in \mathscr{B}_Y$ [3, Proposition 7.26, Lemma 7.28 and Appendix B]. From now on we shall write $f(B \mid \cdot)$ instead of $f(\cdot)(B)$.

Let $X_1, X_2, \ldots$ be a sequence of nonempty sets. The Cartesian products of $X_1, \ldots, X_n$ and $X_1, X_2, \ldots,$ are denoted by $X_1 \cdots X_n$ and $X_1 X_2 \cdots$, respectively. Let $X_1, X_2, \ldots$ be Borel spaces. We assume that the product spaces $X_1 \cdots X_n$ and $X_1 X_2 \cdots$ will have their product topologies and the product σ-algebras. It is well known that the product σ-algebra $\mathscr{B}_{X_1} \cdots \mathscr{B}_{X_n}$ in $X_1 \cdots X_n$ is equal to $\mathscr{B}_{X_1 \cdots X_n}$. A similar result is also valid for the product space $X_1 X_2 \cdots$ (see [3, Proposition 7.13]).

3. The Stochastic Game Model

A *zero-sum discrete-time nonstationary stochastic game* G which we consider is defined by a sequence of objects $\{S_n, X_n, Y_n, A_n, B_n, q_n, u; n \in N\}$ having the following meaning:

(i) S_n is a nonempty Borel space, the *state space* at stage n.

(ii) X_n and Y_n are nonempty Borel spaces, the *action spaces* of players I and II, respectively, at stage n.

Let $H_1 = S_1$, $H_n = S_1 X_1 Y_1 \cdots S_n$, and $H_\infty = S_1 X_1 Y_1 S_2 X_2 Y_2 \cdots$. Then H_n is the set of histories up to stage $n \in N$ while H_∞ is the set of all infinite histories of the game.

Let A_n^0 (B_n^0) be a multifunction from $S_1 \cdots S_n$ into the set of nonempty subsets of X_n (Y_n). For any $h_n = (s_1, x_1, y_1, ..., s_n) \in H_n$, let $A_n(h_n) = A_n^0(s_1, ..., s_n)$ and $B_n(h_n) = B_n^0(s_1, ..., s_n)$.

(iii) $A_n = \{(h_n, x_n): x_n \in A_n(h_n)\}$ and $B_n = \{(h_n, y_n): y_n \in B_n(h_n)\}$. We assume that A_n (B_n) is an analytic (a Borel) subset of $H_n X_n$ ($H_n Y_n$). For each $h_n \in H_n$, $A_n(h_n)$ ($B_n(h_n)$) represents the *set of all admissible actions* for player I (II) under the history $h_n \in H_n$.

For each $(s_1, ..., s_n) \in S_1 \cdots S_n$, $n \in N$, we put

$$A^n(s_1, ..., s_n) = A_1^0(s_1) \cdots A_n^0(s_1, ..., s_n),$$

$$B^n(s_1, ..., s_n) = B_1^0(s_1) \cdots B_n^0(s_1, ..., s_n),$$

$$C_n = \{(h_n, x_n, y_n): h_n \in H_n,\ x_n \in A_n(h_n) \text{ and } y_n \in B_n(h_n)\}.$$

(iv) q_n is a Borel measurable transition probability from $H_n X_n Y_n$ to S_{n+1}. The sequence $\{q_n\}$ constitutes the *transition law* of the game. For a given history $h_n \in H_n$ and actions x_n and y_n chosen by the players at stage n, $q_n(\cdot \mid h_n, x_n, y_n)$ is the conditional distribution of the state at stage $n+1$.

(v) $u: H_\infty \to R^*$ is a bounded below upper semianalytic *payoff function* for player I.

Let $\mathscr{F}_n$ be the set of all universally measurable transition probabilities $f_n: H_n \to P_{X_n}$ such that $f_n(h_n) \in P_{A_n(h_n)}$ for every $h_n \in H_n$. The set $\mathscr{F}_n$ is called the set of *feasible controls* of player I at stage $n \in N$. Similarly, we define the set $\mathscr{G}_n$ of feasible controls of player II at stage $n \in N$.

A universally measurable *strategy* for player I (II) is a sequence $f = \{f_n\}$ ($g = \{g_n\}$), where $f_n \in \mathscr{F}_n$ ($g_n \in \mathscr{G}_n$) for each $n \in N$. We write $\mathscr{F}$ ($\mathscr{G}$) for the set of all strategies for player I (II).

Let ξ_n, α_n, and β_n ($n \in N$) denote the projections from H_∞ on S_n, X_n, and Y_n, respectively. Then the random variables ξ_n, α_n, and β_n describe the state at stage n, and the actions chosen by players I and II, respectively, at stage n. According to the theorem of Ionescu Tulcea, for each pair $(f, g) \in \mathscr{F}\mathscr{G}$ and each initial state $s_1 \in S_1$, the stochastic process $\{(\xi_n, \alpha_n, \beta_n)\}$ has a unique probability measure $P_{fg}(\cdot \mid s_1)$ defined on the product space $X_1 Y_1 S_2 X_2 Y_2 \cdots$ (see [21, p. 162] or [3, Proposition 7.45]). Thus, for each pair $(f, g) \in \mathscr{F}\mathscr{G}$, we define an *expected payoff* to player I in the game G at an initial state $s_1 \in S_1$ to be

$$E(u, f, g)(s_1) = \int u(s_1, h)\, P_{fg}(dh \mid s_1). \tag{3.1}$$

Define, for each $s_1 \in S_1$,

$$L(G)(s_1) = \sup_{f \in \mathscr{F}} \inf_{g \in \mathscr{G}} E(u, f, g)(s_1),$$

and

$$U(G)(s_1) = \inf_{g \in \mathscr{G}} \sup_{f \in \mathscr{F}} E(u, f, g)(s_1).$$

Then $L(G)$ $(U(G))$ is called the *lower* (*upper*) *value function* of the game G. It is always true that $L(G) \leqslant U(G)$. If $L(G) = U(G)$, then this common function is called the *value function* of the game G and is denoted by $V(G)$.

Suppose the value function $V(G)$ exists and define the following set $D = \{s_1 \in S_1 \colon V(G)(s_1) < +\infty\}$. Let $\varepsilon > 0$ be given.

A strategy $\bar{f} \in \mathscr{F}$ is called *ε-optimal* for player I if

$$\inf_{g \in \mathscr{G}} E(u, \bar{f}, g)(s_1) + \varepsilon \geqslant V(G)(s_1) \qquad \text{for all } s_1 \in D,$$

and

$$\inf_{g \in \mathscr{G}} E(u, \bar{f}, g)(s_1) \geqslant 1/\varepsilon \qquad \text{for all } s_1 \in S_1 - D.$$

A strategy $\bar{g} \in \mathscr{G}$ is called *optimal* for player II if

$$\sup_{f \in \mathscr{F}} E(u, f, \bar{g})(s_1) \leqslant V(G)(s_1) \qquad \text{for all } s_1 \in S_1.$$

Our objective in this paper is to present conditions on the initial data (iii)–(v) of the game G that yield the existence of the value function $V(G)$ and optimal or ε-optimal strategies for both players. To make certain assumptions on the payoff u we introduce the following classes $\mathscr{L}_i(H_n)$ of extended real-valued functions on H_n $(i = 1, 2, n \in N)$. By $\mathscr{L}_1(H_1)$ $(\mathscr{L}_2(H_1))$ we mean the set of all u.s.a. (Borel measurable) functions on H_1. A function $w: H_{n+1} \to R^*$ ($n \in N$) belongs to $\mathscr{L}_1(H_{n+1})$ if and only if there exists a nondecreasing sequence $\{w^m\}$ of bounded u.s.a. functions on H_{n+1} such that, for each $m \in N$ and $(s_1, x_1, ..., s_n, x_n, s_{n+1}) \in S_1 X_1 \cdots S_n X_n S_{n+1}$, the function $w^m(s_1, x_1, \cdot, ..., s_n, x_n, \cdot, s_{n+1})$ is continuous on $B^n(s_1, ..., s_n)$ and $w^m \nearrow w$ on $C_n S_{n+1}$ as $m \to \infty$. A function $w: H_{n+1} \to R^*$ ($n \in N$) belongs to $\mathscr{L}_2(H_{n+1})$ if and only if w is Borel measurable and, for each $(s_1, x_1, ..., s_n, x_n, s_{n+1}) \in S_1 X_1 \cdots S_n X_n S_{n+1}$, $w(s_1, x_1, \cdot, ..., s_n, x_n, \cdot, s_{n+1})$ is lower semicontinuous (l.s.c.) on $B^n(s_1, ..., s_n)$ while, for each $(s_1, y_1, ..., s_n, y_n, s_{n+1}) \in S_1 Y_1 \cdots S_n Y_n S_{n+1}$, $w(s_1, \cdot, y_1, ..., s_n, \cdot, y_n, s_{n+1})$ is upper semicontinuous (u.s.c.) on $A^n(s_1, ..., s_n)$.

We now specify two special cases of the game G to be considered in the sequel:

Borel Space Model 1 (BM_1 for short). Here besides (i)–(iv) we assume:

(a) There exists a nondecreasing sequence of bounded functions $u_{m+1} \in \mathscr{L}_1(H_{m+1})$ such that $u_{m+1} \nearrow u$ on H_∞ as $m \to \infty$.

(b) For each $n \in N$, $h_n \in H_n$, the set $B_n(h_n)$ is compact in Y_n.

(c) For each $n \in N$, $(s_1, x_1, ..., s_n, x_n) \in S_1 X_1 \cdots S_n X_n$ and $C \in \mathscr{B}_{S_{n+1}}$, the function $q_n(C \mid s_1, x_1, \cdot, ..., s_n, x_n, \cdot)$ is continuous on $B^n(s_1, ..., s_n)$.

Borel Space Model 2 (BM_2 for short). Here besides (i)–(iv) and assumptions (b) and (c) of BM_1 we impose the following additional conditions:

(a) There exists a nondecreasing sequence of bounded functions $u_{m+1} \in \mathscr{L}_2(H_{m+1})$ such that $u_{m+1} \nearrow u$ on H_∞ as $m \to \infty$.

(b) For each $n \in N$, A_n is a Borel subset of $H_n X_n$ having compact sections $A_n(h_n)$, $h_n \in H_n$.

(c) For each $n \in N$, $(s_1, y_1, ..., s_n, y_n) \in S_1 Y_1 \cdots S_n Y_n$ and $C \in \mathscr{B}_{S_{n+1}}$, the function $q_n(C \mid s_1, \cdot, y_1, ..., s_n, \cdot, y_n)$ is continuous on $A^n(s_1, ..., s_n)$.

To simplify the formulation and derivations in the sequel we employ the following operator terminology. For each $w \in M(H_{n+1})$ and $f_n \in \mathscr{F}_n$, $g_n \in \mathscr{G}_n$, $n \in N$, we define the functions $Q_{f_n g_n} w$, $L_n w$, and $U_n w$ on H_n by

$$(Q_{f_n g_n} w)(h_n) = \iiint w(h_n, x_n, y_n, s_{n+1})\, q_n(ds_{n+1} \mid h_n, x_n, y_n) \times f_n(dx_n \mid h_n)\, g_n(dy_n \mid h_n), \tag{3.2}$$

$$(L_n w)(h_n) = \sup_{f_n \in \mathscr{F}_n} \inf_{g_n \in \mathscr{G}_n} (Q_{f_n g_n} w)(h_n), \tag{3.3}$$

$$(U_n w)(h_n) = \inf_{g_n \in \mathscr{G}_n} \sup_{f_n \in \mathscr{F}_n} (Q_{f_n g_n} w)(h_n). \tag{3.4}$$

If $L_n w = U_n w$ for some $w \in M(H_{n+1})$, this common function will be denoted by $V_n w$.

We have assumed that u is the limit of some nondecreasing sequence of functions u_{m+1}, $m \in N$. Thus from the monotone convergence theorem and [3, Proposition 7.45], we conclude that

$$\begin{aligned} E(u, f, g)(s_1) &= \lim_m E(u_{m+1}, f, g)(s_1) \\ &= \lim_m (Q_{f_1 g_1} \cdots Q_{f_m g_m} u_{m+1})(s_1), \end{aligned} \tag{3.5}$$

where $f = \{f_n\} \in \mathscr{F}$, $g = \{g_n\} \in \mathscr{G}$, and $s_1 \in S_1$. Moreover, from [3, Proposition 7.46], we infer that, for each $f \in \mathscr{F}$, $g \in \mathscr{G}$, $E(u, f, g)$ is a bounded below universally measurable extended real-valued function of the initial state s_1.

Remark 3.1. The game model BM_2 is a special case of BM_1 because, for each $m \in N$, $\mathscr{L}_2(H_{m+1}) \subset \mathscr{L}_1(H_{m+1})$ (see Remark 5.2 below). However, under the assumptions of BM_2 we can prove that the value $V(G)$ of the game G is Borel measurable and optimal (or ε-optimal) strategies for both players in that game can be chosen to be Borel measurable (see Sections 4–6). Such a situation does not take place in BM_1 [5, (45)]. It is also known that the value of a zero-sum stochastic game with a u.s.a. payoff function, which is l.s.c. with respect to the actions of Player II only, need not be universally measurable [20]. This means that the assumption $u_{m+1} \in \mathscr{L}_1(H_{m+1})$ in BM_1(a) cannot be essentially weakened. The convergence condition that we impose in BM_1(a) (and in BM_2(a)) is satisfied in the discounted and positive stochastic games (see [11–14, 17, 24, 31] and the references therein). Furthermore we would like to point out that every bounded below l.s.c. $u: H_\infty \to R^*$ may be represented as the limit of a nondecreasing sequence of bounded and continuous functions $u_{m+1}: H_{m+1} \to R$ (see [29, p. 209] or [30, p. 361]).

Remark 3.2. The game G has already been studied by the author (by means of different methods) under additional assumptions that the state and action spaces are countable or finite sets. In [19], we investigate an alternative framework for zero-sum nonstationary stochastic games in which the payoff u is l.s.c. on H_∞ and each transition probability $q_n: H_n \to P_{S_{n+1}}$ is weakly continuous.

Remark 3.3. Assume here that $S_n = S$, $X_n = X$, and $Y_n = Y$, for some Borel spaces S, X, Y, and for all $n \in N$. A transition law $\{q_n\}$ in G is called *stationary* if there is a Borel measurable transition probability $q: SXY \to P_S$ such that, for each $n \in N$, $h_n = (s_1, x_1, y_1, ..., s_n) \in H_n$, $x_n \in X_n$, and $y_n \in Y_n$, we have $q_n(\cdot \mid h_n, x_n, y_n) = q(\cdot \mid s_n, x_n, y_n)$. A game G with compact metric state space S, with finite action spaces X, Y, and with a stationary transition law was studied by Sengupta [32]. The payoff u in Sengupta's game is assumed to be l.s.c. on H_∞, so his model is a special case of BM_2 (see Remark 3.1 or [32, Remark 3]). Our game G is also an extension of stationary stochastic (or Markov) games in which the transition law is stationary and the payoff u has the form

$$u(h) = \sum_{n=1}^{\infty} \beta^n r(s_n, x_n, y_n),$$

where $h = (s_1, x_1, y_1, ...) \in H_\infty$, $\beta \in [0, 1]$, and $r: SXY \to R$ is a Borel measurable nonnegative function. Starting with Shapley [33] (who has assumed that S, X, Y are finite sets and $\beta < 1$) many researchers have treated such games under some compactness and semicontinuity or continuity assumptions (see [11–14, 17, 24] and their references). The model BM_1 is a direct extension of a stationary stochastic game studied by the author in [17].

4. Main Results

First we consider the *finite horizon games* in which the payoffs are decided in a finite number of stages.

Let $u_{m+1} \in M(H_{m+1})$. For each $n \leqslant m$, we denote by G_n^m a game which has the payoff function $u = u_{m+1}$ and proceeds from an arbitrary history $h_n \in H_n$ until stage m. (The games G_n^m will play an important role in the analysis of the game G.) A strategy for player I (II) in such a game is a finite sequence $f = (f_n, ..., f_m)$ $(g = (g_n, ..., g_m))$, where $f_k \in \mathscr{F}_k$ $(g_k \in \mathscr{G}_k)$, $k = n, ..., m$. The expected payoff to player I corresponding to arbitrary strategies f and g of players I and II, respectively, at a partial history $h_n \in H_n$ is given by

$$E(u_{m+1}, f, g)(h_n) = (Q_{f_n g_n} \cdots Q_{f_m g_m} u_{m+1})(h_n).$$

The value functions $L(G_n^m)$, $U(G_n^m)$, $V(G_n^m)$ and optimal strategies for both players in the game G_n^m are defined as in the game G.

We now arrive at our first main result.

THEOREM 4.1. *Under the assumptions of* BM_i $(i = 1$ *or* $2)$ *the game* G_n^m $(n \leqslant m)$ *with a bounded payoff function* $u_{m+1} \in \mathscr{L}_i(H_{m+1})$ *has a bounded value function* $V(G_n^m)$ *and*

$$V(G_n^m) = V_n \cdots V_m u_{m+1} \in \mathscr{L}_i(H_n).$$

Moreover, in BM_1 (BM_2) *player* II *has an optimal limit* (*Borel*) *measurable strategy, and, for each* $\varepsilon > 0$, *player* I *has an* ε*-optimal limit* (*Borel*) *measurable strategy.*

Let $\{u_{m+1}\}$ be the sequence of functions from the assumption (a) of BM_1 or BM_2. For each $m \in N$, let $G^m = G_1^m$ be an m-stage game corresponding to the payoff function u_{m+1}.

We now can state our second main result.

THEOREM 4.2. *Under the assumptions of* BM_1 (BM_2) *the game* G *has a value function* $V(G)$, *player* II *has an optimal limit* (*Borel*) *measurable strategy, and, for each* $\varepsilon > 0$, *player* I *has an* ε*-optimal limit* (*Borel*) *measurable strategy. Moreover,* $V(G)$ *is upper semianalytic* (*Borel measurable*), *and*

$$V(G) = \lim_m V(G^m).$$

Remark 4.1. Theorem 4.1 generalizes Proposition 2.1 of [15] while Theorem 4.2 extends Theorem 4.1 of [15] and the main result of Sengupta from [32].

5. MINIMAX SELECTION THEOREMS

In this section we establish two measurable selection theorems which are crucial in our development.

THEOREM 5.1. *If the assumptions* (b) *and* (c) *in* BM_1 *hold and* $w \in \mathscr{L}_1(H_{n+1})$, *then*

$$L_n w = U_n w = V_n w \qquad \text{and} \qquad V_n w \in \mathscr{L}_1(H_n).$$

Moreover, there exists a limit measurable control function $\bar{g}_n \in \mathscr{G}_n$ *such that*

$$V_n w = \sup_{f_n \in \mathscr{F}_n} Q_{f_n \bar{g}_n} w, \tag{5.1}$$

and if $w \in \mathscr{L}_1(H_{n+1})$ *is bounded, then, for each* $\varepsilon > 0$, *there exists a limit measurable control function* $\bar{f}_n \in \mathscr{F}_n$ *such that*

$$V_n w \leqslant \inf_{g_n \in \mathscr{G}_n} Q_{\bar{f}_n g_n} w + \varepsilon. \tag{5.2}$$

THEOREM 5.2. *Let assumptions* (b) *and* (c) *in* BM_1 *and* BM_2 *be satisfied. If* $w \in \mathscr{L}_2(H_{n+1})$ *is bounded, then*

$$L_n w = U_n w = V_n w \qquad \text{and} \qquad V_n w \in \mathscr{L}_2(H_n).$$

Moreover, there exists a Borel measurable control function $\bar{f}_n \in \mathscr{F}_n$ *such that*

$$V_n w \leqslant \inf_{g_n \in \mathscr{G}_n} Q_{\bar{f}_n g_n} w. \tag{5.3}$$

If $w\colon H_{n+1} \to R^*$ *is the limit of a nondecreasing sequence* $\{w^m\}$ *of functions from* $\mathscr{L}_2(H_{n+1})$, *then*

$$L_n w = U_n w = V_n w = \lim_m V_n w^m,$$

and there exists a Borel measurable control function $\bar{g}_n \in \mathscr{G}_n$ *such that*

$$V_n w = \sup_{f_n \in \mathscr{F}_n} Q_{f_n \bar{g}_n} w. \tag{5.4}$$

Remark 5.1. Theorem 5.1 generalizes Theorem 5.1 from [17]. For predecessors of Theorem 5.2 (from the theory of stationary stochastic games) consult [24, 13].

We precede the proofs of Theorems 5.1 and 5.2 by a number of auxiliary results.

Let T, X, and Y be Borel spaces. Let A be an analytic subset of TX, and let B be a Borel subset of TY. Define $C = \{(t, x, y)\colon t \in T,\ x \in A(t),$ and $y \in B(t)\}$. Then C is an analytic subset of TXY [18, Lemma 1.1].

The following result is basic for this paper (see [18, Theorem 2.1]).

LEMMA 5.1. *Let* $u\colon C \to R^*$ *be a bounded below function such that*

$$v(t) := \sup_{x \in A(t)} \inf_{y \in B(t)} u(t, x, y) = \inf_{y \in B(t)} \sup_{x \in A(t)} u(t, x, y), \qquad t \in T.$$

If $B(t)$ *is compact, for each* $t \in T$, *and* u *is the limit of a nondecreasing sequence* $\{u_n\}$ *of upper semianalytic functions on* C *such that, for each* $(t, x) \in A$, $n \in N$, $u_n(t, x, \cdot)$ *is continuous on* $B(t)$ *endowed with the relative topology, then*:

(a) v *is upper semianalytic.*

(b) *There is a limit measurable function* $\bar{g}\colon T \to Y$ *such that, for each* $t \in T$, $\bar{g}(t) \in B(t)$ *and*

$$v(t) = \sup_{x \in A(t)} u(t, x, \bar{g}(t)).$$

(c) *If in addition* u *is bounded from above, then for each* $\varepsilon > 0$, *there is a limit measurable function* $\bar{f}\colon T \to X$ *such that, for each* $t \in T$, $\bar{f}(t) \in A(t)$ *and*

$$v(t) \leqslant \inf_{y \in B(t)} u(t, \bar{f}(t), y) + \varepsilon.$$

LEMMA 5.2. *Let S and Y be Borel spaces, w a bounded function from $M(SY)$ such that $w(s, \cdot)$ is continuous on Y, for each $s \in S$. Let q be a Borel measurable transition probability from Y to S such that $q(B \mid \cdot)$ is continuous on Y, for each $B \in \mathscr{B}_S$. Then the function*

$$y \mapsto \int w(s, y)\, q(ds \mid y)$$

is continuous on Y.

Proof. Let $y_n \to y_0$ as $n \to \infty$. Define a probability measure $p \in P_S$ by

$$p(\cdot) = \sum_{m=0}^{\infty} 2^{-m-1} q(\cdot \mid y_m).$$

By [3, Lemma 7.27], for each $m \geqslant 0$, there is a bounded Borel measurable function $w_m: S \to R$ and a Borel subset C_m of S such that $w_m(\cdot) = w(\cdot, y_m)$ on C_m and $p(C_m) = 1$. Let $C = \bigcap_{m=0}^{\infty} C_m$. Then $p(C) = 1$ too. Clearly, $q(C \mid y_m) = 1$ for each $m \geqslant 0$, and $w_m(\cdot) = w(\cdot, y_m)$ on C. Moreover, $w_n \to w_0$ on C, as $n \to \infty$. By the above and [25, Proposition 18 on p. 232], we obtain

$$\begin{aligned} \int_S w(s, y_n)\, q(ds \mid y_n) &= \int_C w_n(s)\, q(ds \mid y_n) \to \int_C w_0(s)\, q(ds \mid y_0) \\ &= \int_C w(s, y_0)\, q(ds \mid y_0) = \int_S w(s, y_0)\, q(ds \mid y_0). \end{aligned}$$

This completes the proof.

Define

$$\begin{aligned} \bar{A}_n &= \{(h_n, p): h_n \in H_n,\ p \in P_{A_n(h_n)}\}, \\ \bar{B}_n &= \{(h_n, r): h_n \in H_n,\ r \in P_{B_n(h_n)}\}, \\ \bar{C}_n &= \{(h_n, p, r): h_n \in H_n,\ p \in P_{A_n(h_n)},\ r \in P_{B_n(h_n)}\}. \end{aligned}$$

We have the following result.

LEMMA 5.3. *Assume* (i)–(iii). *Then*:

(a) *$\bar{A}_n$ is an analytic subset of $H_n P_{X_n}$.*

(b) *$\bar{B}_n$ is a Borel subset of $H_n P_{Y_n}$ and $P_{B_n(h_n)}$ is compact for each $h_n \in H_n$.*

(c) *$\bar{C}_n$ is an analytic subset of $H_n P_{X_n} P_{Y_n}$.*

If in addition A_n is Borel, then:

(d) *$\bar{A}_n$ and $\bar{C}_n$ are Borel sets.*

Proof. This follows from [18, Corollary 4.1; 3, Proposition 7.22]. Define $\bar{A}^1(s_1) = P_{A_1^0(s_1)}$, $\bar{B}^1(s_1) = P_{B_1^0(s_1)}$, and for $n \geqslant 2$,

$$\bar{A}^n(s_1, ..., s_n) = \{(x_1, ..., x_{n-1}, p) : x_k \in A_k^0(s_1, ..., s_k),\\ k = 1, ..., n-1, p \in P_{A_n^0(s_1, ..., s_n)}\},$$

and

$$\bar{B}^n(s_1, ..., s_n) = \{(y_1, ..., y_{n-1}, r) : y_k \in B_k^0(s_1, ..., s_k),\\ k = 1, ..., n-1, r \in P_{B_n^0(s_1, ..., s_n)}\},$$

where $s_k \in S_k$, $k = 1, ..., n$.

In what follows n is an arbitrary positive integer. For any $w \in M(H_{n+1})$ and $(h_n, p, r) \in H_n P_{X_n} P_{Y_n}$, we define

$$K(h_n, p, r)(w) = \iiint w(h_n, x_n, y_n, s_{n+1})\\ \times q_n(ds_{n+1} \mid h_n, x_n, y_n)\, p(dx_n)\, r(dy_n).$$

Note that, for each $h_n \in H_n$,

$$\begin{aligned} (L_n w)(h_n) &= \sup_{p \in P_{A_n(h_n)}} \inf_{r \in P_{B_n(h_n)}} K(h_n, p, r)(w), \\ (U_n w)(h_n) &= \inf_{r \in P_{B_n(h_n)}} \sup_{p \in P_{A_n(h_n)}} K(h_n, p, r)(w). \end{aligned} \tag{5.5}$$

LEMMA 5.4. *Assume* (i), (ii), *and* (iv). *If w is a bounded below upper semianalytic (Borel measurable) extended real-valued function on H_{n+1}, then $K(\cdot, \cdot, \cdot)(w)$ is an upper semianalytic (a Borel measurable) extended real-valued function on $H_n P_{X_n} P_{Y_n}$.*

Proof. If w is bounded below and u.s.a., then the result follows from [3, Proposition 7.48; 5, (32) and (35)]. If w is bounded and Borel measurable, then both the functions w and $-w$ are u.s.a. Thus, both $K(\cdot, \cdot, \cdot)(w)$ and $-K(\cdot, \cdot, \cdot)(w)$ $(= K(\cdot, \cdot, \cdot)(-w))$ are u.s.a., and from Suslin's theorem [10], it follows that $K(\cdot, \cdot, \cdot)(w)$ is Borel measurable. By a standard limiting argument we can now prove that $K(\cdot, \cdot, \cdot)(w)$ is Borel measurable for any w which is bounded from below.

LEMMA 5.5. *Assume* (i)–(iv) *and* BM_1(c). *Let $w \in \mathscr{L}_1(H_{n+1})$ and let $\{w^m\}$ be a sequence from the definition of w. Then:*

(a) *$K(s_1, x_1, \cdot, ..., s_n, p, \cdot)(w^m)$ is continuous on $\bar{B}^n(s_1, ..., s_n)$, for each $(s_1, x_1, ..., s_n, p) \in S_1 X_1 \cdots S_n P_{X_n}$.*

(b) *$K(\cdot, \cdot, \cdot)(w^m) \nearrow K(\cdot, \cdot, \cdot)(w)$ on $\bar{C}_n$ as $m \to \infty$.*

(c) *$K(s_1, x_1, \cdot, ..., s_n, p, \cdot)(w)$ is lower semicontinuous on $\bar{B}^n(s_1, ..., s_n)$, for each $(s_1, x_1, ..., s_n, p) \in S_1 X_1 \cdots S_n P_{X_n}$.*

Proof.

(a) We assume that w^m is bounded on H_{n+1} and moreover $w^m(s_1, x_1, \cdot, ..., s_n, x_n, \cdot, s_{n+1})$ is continuous on $B^n(s_1, ..., s_n)$, for each $m \in N$ and $(s_1, x_1, ..., s_n, x_n, s_{n+1}) \in S_1 X_1 \cdots S_n X_n S_{n+1}$. Thus (a) follows from Lemma 5.2 [26, Lemma 3.4] and the dominated convergence theorem.

(b) Since by assumption $w^m \nearrow w$ on $C_n S_{n+1}$ as $m \to \infty$, so (b) follows from the monotone convergence theorem.

(c) is an immediate consequence of (a) and (b).

LEMMA 5.6. *Assume* (i)–(iii), BM_1(b), *and* BM_2(b). *A bounded function w belongs to $\mathscr{L}_2(H_{n+1})$ if and only if there exist sequences $\{w^m\}$ and $\{u^m\}$ of bounded Borel measurable functions on H_{n+1} such that*:

(a) *$w^m \nearrow w$ and $u^m \searrow w$ on $C_n S_{n+1}$ as $m \to \infty$.*

(b) *For each $(s_1, x_1, ..., s_n, x_n, s_{n+1}) \in S_1 X_1 \cdots S_n X_n S_{n+1}$ and $m \in N$, the function $w^m(s_1, x_1, \cdot, ..., s_n, x_n, \cdot, s_{n+1})$ is continuous on $B^n(s_1, ..., s_n)$.*

(c) *For each $(s_1, y_1, ..., s_n, y_n, s_{n+1}) \in S_1 Y_1 \cdots S_n Y_n S_{n+1}$ and $m \in N$, the function $u^m(s_1, \cdot, y_1, ..., s_n, \cdot, y_n, s_{n+1})$ is continuous on $A^n(s_1, ..., s_n)$.*

Proof. This is a well-known fact. See, for example, [18, pp. 475–476; 6, Theorem 3.10; 28, (4.1) on p. 437].

Remark 5.2. Let $w \in \mathscr{L}_2(H_{n+1})$ be a bounded below function. Applying Lemma 5.6 to $w_n = \min\{w, n\}$ we obtain a nondecreasing sequence $\{w_n^m\}$ of Borel measurable functions such that $w_n^m \nearrow w_n$ on $C_n S_{n+1}$ and $w_n^m(s_1, x_1, \cdot, ..., s_n, x_n, \cdot, s_{n+1})$ is continuous on $B^n(s_1, ..., s_n)$, for each $(s_1, x_1, ..., s_n, x_n, s_{n+1}) \in S_1 X_1 \cdots S_n X_n S_{n+1}$. Note that $w_m^m \nearrow w$ as $m \to \infty$. This implies that $\mathscr{L}_2(H_{n+1}) \subset \mathscr{L}_1(H_{n+1})$. Similarly, we can prove that if w is the limit of a nondecreasing sequence of functions w^m from $\mathscr{L}_2(H_{n+1})$, then w belongs to $\mathscr{L}_1(H_{n+1})$.

LEMMA 5.7. *Assume* (i)–(iv), BM_1(b, c), *and* BM_2(b, c). *Let w be a bounded function from $\mathscr{L}_2(H_{n+1})$. Then*:

(a) *$K(s_1, x_1, \cdot, ..., s_n, p, \cdot)(w)$ is lower semicontinuous on $\bar{B}^n(s_1, ..., s_n)$, for each $(s_1, x_1, ..., s_n, p) \in S_1 X_1 \cdots S_n P_{X_n}$.*

(b) *$K(s_1, \cdot, y_1, ..., s_n, \cdot, r)(w)$ is upper semicontinuous on $\bar{A}^n(s_1, ..., s_n)$, for each $(s_1, y_1, ..., s_n, r) \in S_1 Y_1 \cdots S_n P_{Y_n}$.*

Proof. By Lemma 5.6, $w \in \mathscr{L}_1(H_{n+1})$. Thus, (a) follows from Lemma 5.5(c). A proof of (b) can be given by a translation of that of Lemma 5.5, using the sequence $\{u^m\}$ from Lemma 5.6.

LEMMA 5.8. *Let* $\mathrm{BM}_1(\mathrm{b}, \mathrm{c})$ *be satisfied and let $w \in \mathscr{L}_1(H_{n+1})$. Then*

$$L_n w = U_n w = V_n w.$$

Proof. By Lemma 5.5(c), for each $(h_n, p) \in H_n P_{X_n}$, $K(h_n, p, \cdot)(w)$ is l.s.c. on $P_{B_n(h_n)}$ being a compact and convex subset of P_{Y_n}. The result now follows from (5.5) and the well-known minimax theorem of Ky Fan [8, Theorem 2].

Finally, we give the following lemma.

LEMMA 5.9. *Let X be a separable metric space and let Y be a compact metric space. Assume that $\{u_n\}$ is a nondecreasing sequence of bounded below Borel measurable functions $u_n: XY \to R$ such that, for each $n \in N$, $x \in X$, $u_n(x, \cdot)$ is lower semicontinuous on Y. Define*

$$\bar{u}_n(p, r) = \iint u_n(x, y)\, p(dx)\, r(dy), \qquad p \in P_X, r \in P_Y.$$

Then

$$\lim_n \inf_{r \in P_Y} \sup_{p \in P_X} \bar{u}_n(p, r) = \inf_{r \in P_Y} \sup_{p \in P_X} \lim_n \bar{u}_n(p, r).$$

Proof. This follows from the compactness of P_Y, the lower semicontinuity of $\bar{u}_n(p, \cdot)$, and [27, Proposition 10.1], see [24].

Proof of Theorem 5.1. The value function $V_n w$ exists by Lemma 5.8. Let $\{w^m\}$ be a sequence from the definition of $w \in \mathscr{L}_1(H_{n+1})$. By Lemmas 5.3 and 5.4, every function $K(\cdot, \cdot, \cdot)(w^m)$ is u.s.a. on $\bar{C}_n$ being an analytic subset of the Borel space $H_n P_{X_n} P_{Y_n}$. Moreover, from Lemma 5.5(a, b), it follows that $K(\cdot, \cdot, \cdot)(w^m) \nearrow K(\cdot, \cdot, \cdot)(w)$ on $\bar{C}_n$ as $m \to \infty$, and $K(h_n, p, \cdot)(w^m)$ is continuous on $P_{B_n(h_n)}$, for each $m \in N$, $h_n \in H_n$, $p \in P_{A_n(h_n)}$. Keeping all these facts and Lemma 5.3 in mind and using Lemma 5.1, we obtain a limit measurable $\bar{g}_n \in \mathscr{G}_n$ satisfying (5.1), and when w is bounded, then, for each $\varepsilon > 0$, we get a limit measurable $\bar{f}_n \in \mathscr{F}_n$ that satisfies (5.2). Moreover, from Lemma 5.1 we infer that $V_n w$ is u.s.a. on H_n. Thus, it remains to prove that $V_n w \in \mathscr{L}_1(H_n)$ for each $n \in N$. It has already been shown for $n = 1$ ($\mathscr{L}_1(H_1)$ is the set of u.s.a. functions). We now prove that

$V_2 w \in \mathscr{L}_1(H_2)$ for each $w \in \mathscr{L}_1(H_3)$. One can easily see that the proof of the general case $V_n w \in \mathscr{L}_1(H_n)$, $n \geqslant 2$, proceeds along similar lines.

Let $w \in \mathscr{L}_1(H_3)$. Then there is a nondecreasing sequence $\{w^m\}$ of u.s.a. functions on H_3 such that $w^m(s_1, x_1, \cdot, s_2, x_2, \cdot, s_3)$ is continuous on $B^2(s_1, s_2)$, for each $(s_1, x_1, s_2, x_2, s_3)$, $m \in N$, and $w^m \nearrow w$ on $C_2 S_3$ as $m \to \infty$.

Let ρ be a metric on $Y_1 P_{Y_2}$ equivalent to the product topology on $Y_1 P_{Y_2}$. Define

$$\begin{aligned}
&\Phi_{mn}(s_1, x_1, b_1, y_1, s_2, x_2, r, t)\\
&\qquad = K(s_1, x_1, b_1, s_2, p_{x_2}, t)(w^m) + n\rho[(y_1, r), (b_1, t)],
\end{aligned}$$

where $(s_1, x_1, y_1, s_2) \in H_2$, $b_1 \in Y_1$, $x_2 \in X_2$, r, $t \in P_{Y_2}$, and p_{x_2} is the probability measure on $\mathscr{B}_{X_2}$ which assigns unit point mass to x_2. (The mapping $x_2 \mapsto p_{x_2}$ is a homeomorphism [3, Corollary 7.21.1].)

Let

$$\varphi_{mn}(s_1, x_1, y_1, s_2, x_2, r) = \inf_{(b_1, t) \in \bar{B}^2(s_1, s_2)} \Phi_{mn}(s_1, x_1, b_1, y_1, s_2, x_2, r, t),$$

$m, n \in N$, $(s_1, x_1, y_1, s_2) \in H_2$, $x_2 \in X_2$, and $r \in P_{Y_2}$.

By the proof of the theorem of Baire [1, p. 390] and Lemma 5.5(a), we have

$$\varphi_{mn}(s_1, x_1, y_1, s_2, x_2, r) \nearrow K(s_1, x_1, y_1, s_2, p_{x_2}, r)(w^m) \tag{5.6}$$

as $n \to \infty$, for each $m \in N$, $(s_1, x_1, s_2, x_2) \in S_1 X_1 S_2 X_2$, and $(y_1, r) \in \bar{B}^2(s_1, s_2)$.

Recall that B_1 is a Borel subset of $H_1 Y_1$ with compact sections $B_1(h_1)$. By [17, F2.1], there exists a sequence of Borel measurable functions $g_k^1\colon H_1 \to Y_1$, $k \in N$, such that $\{g_k^1(h_1)\}$ is dense in $B_1(h_1)$, for each $h_1 \in H_1$. By Lemma 5.3(b) and [17, F2.1], there exists a sequence of Borel measurable functions $g_l^2\colon H_2 \to P_{Y_2}$, $l \in N$, such that $\{g_l^2(h_2)\}$ is dense in $P_{B_2(h_2)}$, for each $h_2 \in H_2$. Using these sequences and Lemma 5.5(a), we infer that

$$\begin{aligned}
&\varphi_{mn}(s_1, x_1, y_1, s_2, x_2, r)\\
&\qquad = \inf_{k, l \in N} \Phi_{mn}(s_1, x_1, g_k^1(s_1), y_1, s_2, x_2, r, g_l^2(s_1, x_1, y_1, s_2)),
\end{aligned} \tag{5.7}$$

where $s_1 = h_1 \in S_1$, $(s_1, x_1, y_1, s_2) \in H_2$, $x_2 \in X_2$, $r \in P_{Y_2}$, and $m, n \in N$. Since $K(\cdot, \cdot, \cdot)(w^m)$ is u.s.a. on $H_2 P_{X_2} P_{Y_2}$ (Lemma 5.4) and g_k^1, g_l^2 are Borel measurable, so from (5.7), we conclude that φ_{mn} is u.s.a. on $H_2 X_2 P_{Y_2}$.

Define

$$M_{mn}(h_2, r) = \sup_{x_2 \in A_2(h_2)} \varphi_{mn}(h_2, x_2, r), \qquad h_2 \in H_2, r \in P_{Y_2}.$$

By [3, Proposition 7.47], M_{mn} is u.s.a. on $H_2 P_{Y_2}$. (A_2 is assumed to be an analytic subset of $H_2 X_2$.)

Recall that $A_2(s_1, x_1, y_1, s_2) = A_2^0(s_1, s_2)$ for each $(s_1, x_1, y_1, s_2) \in H_2$. Then note that, for each $m, n \in N$, (s_1, x_1, y_1, s_2, r) and $(s_1, x_1, y_1', s_2, r')$, we have

$$\begin{aligned} &|M_{mn}(s_1, x_1, y_1, s_2, r) - M_{mn}(s_1, x_1, y_1', s_2, r')| \\ &\quad \leqslant \sup_{x_2 \in A_2^0(s_1, s_2)} \sup_{(b_1, t) \in \bar{B}^2(s_1, s_2)} |\Phi_{mn}(s_1, x_1, b_1, y_1, s_2, x_2, r, t) \\ &\qquad - \Phi_{mn}(s_1, x_1, b_1, y_1', s_2, x_2, r', t)| \leqslant n\rho[(y_1, r), (y_1', r')]. \end{aligned}$$

This implies that $M_{mn}(s_1, x_1, \cdot, s_2, \cdot)$ is continuous on $Y_1 P_{Y_2}$, for each $m, n \in N$.

Define, for $m, n \in N$ and $h_2 = (s_1, x_1, y_1, s_2) \in H_2$,

$$\Psi_{mn}(s_1, x_1, y_1, s_2) = \inf_{r \in P_{B_2^0(s_1, s_2)}} M_{mn}(s_1, x_1, y_1, s_2, r). \tag{5.8}$$

Since $B_2^0(s_1, s_2) = B_2(s_1, x_1, y_1, s_2)$, for each $h_2 = (s_1, x_1, y_1, s_2) \in H_2$, so

$$\Psi_{mn}(h_2) = \inf_{r \in P_{B_2(h_2)}} M_{mn}(h_2, r), \qquad h_2 \in H_2. \tag{5.9}$$

By Lemma 5.3(b), $\bar{B}_2$ is a Borel subset of $H_2 P_{Y_2}$, and $P_{B_2(h_2)}$ is compact for each $h_2 \in H_2$. At the same time M_{mn} is u.s.a. on $\bar{B}_2$ and $M_{mn}(h_2, \cdot)$ is continuous on $P_{B_2(h_2)}$, $h_2 \in H_2$. Thus, from [16, Proposition 3.2] and (5.9), we conclude that Ψ_{mn} is u.s.a. on H_2, for every $m, n \in N$. From (5.8) and Berge's theorem [2, p. 122], the compactness of $P_{B_2(h_2)}$, $h_2 \in H_2$, and the continuity of $M_{mn}(s_1, x_1, \cdot, s_2, \cdot)$, it follows that $\Psi_{mn}(s_1, x_1, \cdot, s_2)$ is continuous on Y_1, for each $(s_1, x_1, s_2) \in S_1 X_1 S_2$.

Now, we shall show that $\Psi_{nn} \nearrow V_2 w$ on $C_1 S_2$ as $n \to \infty$. This implies that $V_2 w \in \mathscr{L}_1(H_2)$. For each $m \leqslant n$, since $\{w^m\}$ is nondecreasing and $w^n \leqslant w$, we have

$$M_{mn}(h_2, r) \leqslant M_{nn}(h_2, r) \leqslant \sup_{x_2 \in A_2(h_2)} K(h_2, p_{x_2}, r)(w), \tag{5.10}$$

where $h_2 \in C_1 S_2$, $r \in P_{B_2(h_2)}$. From (5.6), we conclude that

$$\lim_n M_{mn}(h_2, r) = \sup_{x_2 \in A_2(h_2)} K(h_2, p_{x_2}, r)(w^m),$$

for each $m \in N$. This and the monotone convergence theorem imply that

$$\begin{aligned}\lim_m \lim_n M_{mn}(h_2, r) &= \lim_m \sup_{x_2 \in A_2(h_2)} K(h_2, p_{x_2}, r)(w^m) \\ &= \sup_{x_2 \in A_2(h_2)} K(h_2, p_{x_2}, r)(w) \qquad (5.11)\end{aligned}$$

for $h_2 \in C_1 S_2$, $r \in P_{B_2(h_2)}$. By [27, Proposition 10.1], the compactness of $P_{B_2(h_2)}$, the continuity of $M_{mn}(h_2, \cdot)$, and (5.11), we obtain

$$\begin{aligned}\lim_m \lim_n \Psi_{mn}(h_2) &= \inf_{r \in P_{B_2(h_2)}} \lim_m \lim_n M_{mn}(h_2, r) \\ &= \inf_{r \in P_{B_2(h_2)}} \sup_{x_2 \in A_2(h_2)} K(h_2, p_{x_2}, r)(w), \\ &\qquad \text{where} \quad h_2 \in C_1 S_2. \qquad (5.12)\end{aligned}$$

But the right-hand side of (5.12) is equal to $(V_2 w)(h_2)$ (see (5.5) and Lemma 5.8). Thus,

$$\lim_m \lim_n \Psi_{mn} = V_2 w \qquad \text{on } C_1 S_2. \qquad (5.13)$$

By (5.10), we have

$$\lim_n \Psi_{mn} \leqslant \lim_n \Psi_{nn} \leqslant V_2 w \text{ on } C_1 S_2, \qquad \text{for all } m \in N.$$

Hence

$$\lim_m \lim_n \Psi_{mn} \leqslant \lim_n \Psi_{nn} \leqslant V_2 w \qquad \text{on } C_1 S_2. \qquad (5.14)$$

Combining (5.13) and (5.14) yields $V_2 w = \lim_n \Psi_{nn}$ on $C_1 S_2$, which completes the proof.

Proof of Theorem 5.2. First of all note that $\bar{A}_n$, $\bar{B}_n$, and $\bar{C}_n$ are Borel sets (Lemma 5.3(b, d)) and $P_{A_n(h_n)}$, $P_{B_n(h_n)}$ are compact sets, for each $h_n \in H_n$.

Let $w \in \mathscr{L}_2(H_{n+1})$ be bounded. Define

$$M(h_n, p) = \inf_{r \in P_{B_n(h_n)}} K(h_n, p, r)(w), \qquad (h_n, p) \in \bar{A}_n.$$

By Lemma 5.4, $K(\cdot, \cdot, \cdot)(w)$ is Borel measurable on $\bar{C}_n$, and from Lemma 5.7, it follows that $K(h_n, p, \cdot)(w)$ is l.s.c. on $P_{B_n(h_n)}$, for each $(h_n, p) \in \bar{A}_n$, and $K(h_n, \cdot, r)(w)$ is u.s.c. on $P_{A_n(h_n)}$, for each $(h_n, r) \in \bar{B}_n$. Using [6, Corollary 1], we infer that M is Borel measurable on $\bar{A}_n$.

Moreover, $M(h_n, \cdot)$ is u.s.c. on $P_{A_n(h_n)}$, $h_n \in H_n$. By (5.5), Remark 5.2, and Lemma 5.8, we have

$$(V_n w)(h_n) = \sup_{p \in P_{A_n(h_n)}} M(h_n, p), \qquad h_n \in H_n. \tag{5.15}$$

Again using [6, Corollary 1], we conclude that $V_n w$ is Borel measurable, and, moreover, there is a Borel measurable $\bar{f}_n \in \mathscr{F}_n$ such that

$$M(h_n, \bar{f}_n(h_n)) = \sup_{p \in P_{A_n(h_n)}} M(h_n, p), \qquad h_n \in H_n.$$

This and (5.15) give (5.3). The fact $V_n w \in \mathscr{L}_2(H_{n+1})$ follows from (5.5), (5.15), the compactness of $\bar{A}^n(s_1, ..., s_n)$ and $\bar{B}^n(s_1, ..., s_n)$, Lemma 5.7, and Berge's theorem [2, p. 122].

Let $w: H_{n+1} \to R^*$ be the limit of a nondecreasing sequence of bounded functions $w^m \in \mathscr{L}_2(H_{n+1})$. Of course $V_n w^m$ $(V_n w)$ exists and is equal to $U_n w^m$ $(U_n w)$ (see Remark 5.2 and Lemma 5.8). Using Lemmas 5.7 and 5.9 and the monotone convergence theorem one can easily show that

$$V_n w = U_n w = U_n \lim_m w^m = \lim_m U_n w^m = \lim_m V_n w^m.$$

Define

$$T_m(h_n, r) = \sup_{p \in P_{A_n(h_n)}} K(h_n, p, r)(w^m),$$

and

$$T(h_n, r) = \sup_{p \in P_{A_n(h_n)}} K(h_n, p, r)(w),$$

where $m \in N$, $(h_n, r) \in \bar{B}_n$. By [6, Corollary 1], T_m is Borel measurable on $\bar{B}_n$. Moreover, for every $h_n \in H_n$, $T_m(h_n, \cdot)$ is l.s.c. on $P_{B_n(h_n)}$ being a compact subset of P_{Y_n}. Since $w^m \nearrow w$ as $m \to \infty$, so $T_m \nearrow T$ on $\bar{B}_n$. Thus, T is Borel measurable, and $T(h_n, \cdot)$ is l.s.c. on $P_{B_n(h_n)}$, $h_n \in H_n$. Using [6, Corollary 1], we obtain a Borel measurable $\bar{g}_n \in \mathscr{G}_n$ such that

$$T(h_n, \bar{g}_n(h_n)) = \inf_{r \in P_{B_n(h_n)}} T(h_n, r), \qquad h_n \in H_n.$$

This, (5.5) and Lemma 5.8 give (5.4). The proof is finished.

6. Proofs of the Main Results

The main results of this paper are based on iterating Theorems 5.1 and 5.2.

Proof of Theorem 4.1. The proof is based on Theorems 5.1 and 5.2 and proceeds by induction, along similar lines as that of Theorem 4.1 in [19], or Proposition 2.1 in [15].

We now return to the infinite horizon game G with the payoff function $u: H_\infty \to R^*$. The proof of Theorem 4.2 is based on Theorem 4.1 and is similar to that of Theorem 4.2 in [19], but to avoid any confusion we give some of its steps.

In the sequel, let $\{u_{m+1}\}$ be a nondecreasing sequence of bounded functions $u_{m+1} \in \mathscr{L}_1(H_{m+1})$ such that $u_{m+1} \nearrow u$ on H_∞ as $m \to \infty$. By Theorem 4.1, each game G_n^m with payoff u_{m+1} has a value function $V(G_n^m)$ and $V(G_n^m) \in \mathscr{L}_1(H_n)$ (here $n \leqslant m$). Clearly, for each $n \in N$, the sequence $\{V(G_n^m)\}$, $m \geqslant n$, is nondecreasing, because so is $\{u_{m+1}\}$. Thus, for each $n \in N$, we may define

$$W_n = \lim_m V(G_n^m).$$

The following fact is easy to prove (see [3, Lemma 7.30(2)]).

LEMMA 6.1. *For each $n \in N$, $W_n \in \mathscr{L}_1(H_n)$.*

LEMMA 6.2. *For each $n \in N$, $W_n = V_n W_{n+1}$.*

Proof. By Theorem 4.1, for each $m \geqslant n+1$, we have $V(G_n^m) = V_n G_{n+1}^m$, and $V(G_{n+1}^m) \in \mathscr{L}_1(H_{n+1})$. From Lemmas 6.1, 5.5(c), and 5.9, the compactness of $P_{B_n(h_n)}$, $h_n \in H_n$, and the monotone convergence theorem, it follows that

$$\begin{aligned} W_n &= \lim_m V(G_n^m) = \lim_m V_n V(G_{n+1}^m) = \lim_m U_n V(G_{n+1}^m) \\ &= U_n \lim_m V(G_{n+1}^m) = U_n W_{n+1}. \end{aligned}$$

By Lemmas 5.8 and 6.1, $U_n W_{n+1} = V_n W_{n+1}$, which completes the proof.

Recall that $G^m = G_1^m$, and $V(G^m) = L(G^m)$, for each $m \in N$. Because $u_m \leqslant u$ on H_∞, for all $m \in N$, so we have

$$W_1 = \lim_m V(G^m) \leqslant L(G) \text{ on } H_1 \qquad (H_1 = S_1). \tag{6.1}$$

LEMMA 6.3. *For every $k < n$, we have $u_{k+1} \leqslant W_{n+1}$.*

Proof. Let $k < n$. Then we have

$$u_{k+1}(h_{k+1}) \leqslant u_{n+1}(h_{k+1}, h) \leqslant u_{n+2}(h_{k+1}, h, x_{n+1}, y_{n+1}, s_{n+2}),$$

for each $h_{k+1} \in H_{k+1}$, $h \in X_{k+1} Y_{k+1} S_{k+2} \cdots S_{n+1}$, and for every $(x_{n+1}, y_{n+1}, s_{n+2}) \in X_{n+1} Y_{n+1} S_{n+2}$. Hence, it follows that $u_{k+1} \leqslant V(G_{n+1}^{n+1})$. But $V(G_{n+1}^{n+1}) \leqslant V(G_{n+1}^{m})$, for each $m \geqslant n+1$. Thus $u_{k+1} \leqslant \lim_m V(G_{n+1}^m) = W_{n+1}$.

Proof of Theorem 4.2. (a) *Case* BM_1. We know from Lemma 6.1 that $W_n \in \mathscr{L}_1(H_n)$. By Theorem 5.1, for each $n \in N$, there is a limit measurable $\bar{g}_n \in \mathscr{G}_n$ such that

$$V_n W_{n+1} = \sup_{f_n \in \mathscr{F}_n} Q_{f_n \bar{g}_n} W_{n+1}. \tag{6.2}$$

Let $U_{\bar{g}_n} W_{n+1}$ be the right-hand side of (6.2). Let $f = \{f_n\}$ be an arbitrary strategy for player I, and let $k \in N$. By (6.2) and Lemma 6.2, for each $n \in N$, we obtain

$$\begin{aligned} W_1 = V_1 \cdots V_n W_{n+1} &= U_{\bar{g}_1} \cdots U_{\bar{g}_n} W_{n+1} \\ &\geqslant Q_{f_1 \bar{g}_1} \cdots Q_{f_n \bar{g}_n} W_{n+1}. \end{aligned}$$

Taking now $n > k$ and using Lemma 6.3 and (3.1) we obtain

$$W_1 \geqslant Q_{f_1 \bar{g}_1} \cdots Q_{f_n \bar{g}_n} u_{k+1} = E(u_{k+1}, f, \bar{g}),$$

where $\bar{g} = \{\bar{g}_n\} \in \mathscr{G}$, and $f = \{f_n\} \in \mathscr{F}$ is arbitrary. This and (3.5) imply

$$W_1 \geqslant \lim_k E(u_{k+1}, f, \bar{g}) = E(u, f, \bar{g}),$$

for each $f \in \mathscr{F}$. Thus

$$W_1 \geqslant \sup_{f \in \mathscr{F}} E(u, f, \bar{g}) \geqslant U(G). \tag{6.3}$$

Combining (6.1) and (6.3), we conclude that the game G has a value $V(G)$ and $V(G) = \lim_m V(G^m)$. This and [3, Lemma 7.30(2)] imply that $V(G) \in \mathscr{L}_1(H_1)$. Moreover, $\bar{g}$ is an optimal strategy for player II.

Let $\varepsilon > 0$ be given. By Theorem 4.1, for each $m \in N$, player I has a limit measurable $(\varepsilon/2)$-optimal strategy in the m-stage game G^m. Using this and the fact that $V(G^m) \nearrow V(G)$ as $m \to \infty$, one can construct an ε-optimal limit measurable strategy for player I in the game G (see the proof of Theorem 4.2 in [19]).

(b) *Case* BM_2. A proof in this case can be given by a similar manner as that of BM_1, using Theorem 5.2 (especially (5.4)) instead of Theorem 5.1. We only note here that $V(G_n^m) \in \mathscr{L}_2(H_n)$, $m \geqslant n$, and W_n is Borel measurable, and, by Remark 5.2, $W_n \in \mathscr{L}_1(H_n)$, $n \in N$, so Lemma 6.2 applies.

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